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Cartan Matrices and Grothendieck Groups of Stable Categories

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Let A be a finite dimensional algebra over a field K and $\hat{A}$ its repetitive algebra [4]. In [2] Happel proved that if gl. dim $A < \infty$ then there is an equivalence between the derived category $D^b(A)$ and projectively stable module catagory $\underline{\text{mod}}\ \hat{A}$ such that the triangulations of $D^b(A)$ and $\underline{\text{mod}}\ \hat{A}$ are retained by the equivalence.

This investigation is motivated by the converse problem of his theorem. If there is a triangulated category equivalence G: $D^b(A) \simeq \underline{\text{mod}}\ \hat{A}$, G induces naturally the isomorphism between their Grothedieck groups: $K_0(D^b(A)) \simeq K_0\ (\underline{\text{mod}}\ \hat{A})$. As $K_0(D^b(A)) \simeq K_0(A)$ it follows that $K_0(A) \simeq K_0\ (\underline{\text{mod}}\ \hat{A})$ and consequently $K_0\ (\underline{\text{mod}}\ \hat{A})$ is isomorphic to the free abelian group $\mathbb{Z}^{(n)}$ of rank n, where n is the number of non-isomorphic simple A-modules.

The main purpose of this paper is to give a characterization of algebras A such that $K_0\ (\underline{\text{mod}}\ \hat{A}) \simeq K_0(A)$. In Section 2 we prove that in order to hold $K_0\ (\underline{\text{mod}}\ \hat{A}) \simeq K_0(A)$ it is necessary and sufficient that the determinant of the Cartan matrix of A be equal to ± 1. By this result we can easily find some algebras A such that $D^b(A) \not\simeq \underline{\text{mod}}\ \hat{A}$.

1. GROTHENDIECK GROUP $K_0\ (\underline{\text{mod}}\ \hat{A})$

Let B be a Frobenius K-algebra. That is, B is a basic K-algebra and is a direct sum of (not necessarily finite number of) indecomposable right

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ideals which are injective. It is known that, by using the loop space functor Ω^{-1} of Heller as the translation functor, $\underline{\mathrm{mod}}\, B$ can be seen as a triangulated category.

Let X and Y be non-projective B-modules.

For any $f \in \mathrm{Hom}_B(X, Y)$, $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Omega^{-1}X$ is a triangle in $\underline{\mathrm{mod}}\, B$ if Z is the pushout of f and the inclusion ι_X of X into the injective envelope $I(X)$ and if g and h appear in the diagram

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r, "\iota_X"] \arrow[d, "f"'] & I(X) \arrow[r] \arrow[d] & \Omega^{-1} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & Y \arrow[r, "g"] & Z \arrow[r, "h"] & \Omega^{-1}X \arrow[r] & 0
\end{tikzcd}$$

We shall denote by $[X]$, $[Y]$, and $[Z]$ (resp. $[\underline{X}]$, $[\underline{Y}]$, and $[\underline{Z}]$) the isomorphism classes of X, Y, and Z in mod B (resp. $\underline{\mathrm{mod}}\, B$). Let F be the free abelian group generated by the isomorphism classes of modules in $\underline{\mathrm{mod}}\, B$ and F_0 the subgroup generated by $[\underline{X}] - [\underline{Y}] + [\underline{Z}]$ for all triangles $X \to Y \to Z \to \Omega^{-1}X$ as above. Then the Grothendieck group K_0 ($\underline{\mathrm{mod}}\, B$) is defined to be the factor group F/F_0. For the details we refer to [3, pp. 10–23, 95–102].

Now we have

PROPOSITION 1.. *Let B be a Frobenius K-algebra. Then it holds that K_0 ($\underline{\mathrm{mod}}\, B$) $\simeq K_0(B)/\langle[\mathrm{proj}]\rangle$, where $\langle[\mathrm{proj}]\rangle$ is the subgroup* of $K_0(B)$ *generated by the isomorphism classes of projective B-modules.*

Proof. Let us define a homomorphism $\rho'\colon K_0(B) \to K_0$ ($\underline{\mathrm{mod}}\, B$) by $[X] \to [\underline{X}]$ for $X \in \mathrm{mod}\, B$. First, we should check that ρ' is well defined. Let $0 \to X \xrightarrow{u} Y \to Z \to 0$ be an exact sequence in mod B, i.e., $[X] + [Z] = [Y]$ in $K_0(B)$. Taking C_u as the pushout of u and ι_X we have the following commutative diagram with exact rows and columns:

$$\begin{tikzcd}
X \arrow[r, "\iota_X"] \arrow[d, "u"'] & I(X) \arrow[r] \arrow[d, "v"'] & \Omega^{-1}X \arrow[d] \\
Y \arrow[r, "g_u"] \arrow[d] & C_u \arrow[r, "f_u"] \arrow[d] & \Omega^{-1}X \\
Z \arrow[r, phantom, "\simeq"] & Z &
\end{tikzcd}$$

and $X \xrightarrow{u} Y \xrightarrow{g_u} C_u \xrightarrow{f_u} \Omega^{-1}X$ is a triangle in $\underline{\mathrm{mod}}\, B$. Since $C_u \simeq I(X) \oplus Z$ and hence $[\underline{X}] + [\underline{C}_u] = [\underline{Y}]$ as $I(X)$ is projective in mod B. This implies ρ' is well defined.

It is clear that ρ' carries any finitely generated projective B-module to

$[\underline{0}]$ in K_0 ($\underline{\mathrm{mod}}\, B$). So ρ' induces naturally a homomorphism $\rho\colon K_0(B)/\langle[\mathrm{proj}]\rangle \to K_0$ ($\underline{\mathrm{mod}}\, B$].

Conversely we can define a homomorphism $\nu\colon K_0$ ($\underline{\mathrm{mod}}\, B) \to K_0(B)/\langle[\mathrm{proj}]\rangle$ by putting $\nu([\underline{X}]) = [X]$ modulo $\langle[\mathrm{proj}]\rangle$.

Assume $\underline{X} \xrightarrow{f} \underline{Y} \to \underline{C} \to \underline{\Omega}^{-1}\underline{X}$ is a triangle, i.e., $[\underline{X}] + [\underline{C}] = [\underline{Y}]$ in $\underline{\mathrm{mod}}\, B$. Then C is isomorphic to a pushout of f and $\iota_X\colon X \to I(X)$ and the sequence $0 \to X \to I(X) \oplus Y \to C \to 0$ is exact. Hence we have $[X] + [C] = [Y]$ modulo $\langle[\mathrm{proj}]\rangle$ since B is selfinjective. This implies that ν is also well defined.

Now it is immediate that $\nu\rho = \mathrm{id}_{K_0(B)/\langle[\mathrm{proj}]\rangle}$ and $\rho\nu = \mathrm{id}_{K_0(\underline{\mathrm{mod}}\, B)}$ by the above argument.

For a finite dimensional K-algebra A and an A-bimodule M we shall denote by $c_{ij}(M)$ the composition length of right e_jAe_j-module e_iMe_j, where e_i and e_j, $i, j = 1, 2, ..., n$, are primitive idempotents such that $A = \bigoplus_{i=1}^n Ae_i$ is a direct sum decomposition of A into left primitive (i.e., indecomposable) ideals. Then it is immediate that

$$c_{ij}(D(M))(\bar{e}_j\bar{A}\bar{e}_j : K) = c_{ji}(M)(\bar{e}_i\bar{A}\bar{e}_i : K),$$

where $D(M) = \mathrm{Hom}_K(M, K)$, $\bar{A} = A/J(A)$, an $J(A)$ is the Jacobson radical of A. Especially we have

LEMMA 1.2. *$c_{ji}(A)f_i = c_{ij}(D(A))f_j$, where $f_i = (\bar{e}_i\bar{A}\bar{e}_i : K)$ and $f_j = (\bar{e}_j\bar{A}\bar{e}_j : K)$.*

Here we mention that the matrix $(c_{ij}(A))_{ij}$ is the Cartan matrix of A.

For a finite dimensional K-algebra A, the repetitive algebra $\hat{A}$ is defined to be an infinite dimensional K-algebra whose underlying vector K-space is

$$\hat{A} = \Big(\bigoplus_{p\in\mathbb{Z}} A_p\Big) \oplus \Big(\bigoplus_{p\in\mathbb{Z}} D(A)_p\Big),$$

where

$$_{A_p}D(A)_{p\;A_{p+1}} = {}_{A_p}\mathrm{Hom}_K(A, K)_{A_{p+1}},$$

$A_p \simeq A$, and the multiplication is defined by

$$(a_p, \alpha_p)(b_q, \beta_q) = (\delta_{pq}a_pb_q, \delta_{pq}a_p\beta_q + \delta_{p+1,q}\alpha_pb_q),$$

where δ means the Kronecker δ.

Corresponding to a direct sum decomposition $A = \bigoplus_{i=1}^n Ae_i$ of A into primitive left ideals we have the direct sum decomposition of $\hat{A}$ such that

$$\hat{A} = \bigoplus_{p\in\mathbb{Z}} \bigoplus_{i=1}^{n} \hat{A}e_{p,i} = \bigoplus_{p\in\mathbb{Z}} \bigoplus_{i=1}^{n} e_{p,i}\hat{A},$$

where $e_{p,i} \in A_p$ and $_{A_p}\hat{A}e_{p,i} \simeq {}_A Ae_i$, $e_{p,i}\hat{A}_{A_p} \simeq e_iA_A$.

Since the Jacobson radical $J(\hat{A}) = (\bigoplus_{p \in \mathbb{Z}} J(A)_p) \oplus (\bigoplus_{p \in \mathbb{Z}} D(A)_p)$, $\bar{e}_{p,i}\bar{A}_{A_p} \simeq \bar{e}_i\bar{A}_A$, and $\{[\bar{e}_{p,i}\bar{A}] \mid p \in \mathbb{Z},\ i = 1, 2, ..., n\}$ is a free $\mathbb{Z}$-basis of $K_0(\hat{A})$.

Now T: $[\bar{e}_{p,i}\bar{A}] \to [\bar{e}_{p+1,i}\bar{A}]$ defines the Nakayama permutation of $\hat{A}$ because $\bar{e}_{p,i}\bar{A} \simeq$ top of $e_{p,i}\hat{A}$ and $\bar{e}_{p+1,i}\bar{A} \simeq$ socle of $e_{p,i}\hat{A}$. Then we have

$$\begin{aligned} K_0(\hat{A}) &= \bigoplus_{i=1}^{n} \left(\bigoplus_{p \in \mathbb{Z}} \mathbb{Z}[\bar{e}_{p,i}\bar{A}] \right) \\ &= \bigoplus_{i=1}^{n} \left(\bigoplus_{p \in \mathbb{Z}} \mathbb{Z}T^p[\bar{e}_{0,i}\bar{A}] \right) \\ &\simeq \bigoplus_{i=1}^{n} \mathbb{Z}[T, T^{-1}] \\ &= \mathbb{Z}[T, T^{-1}]^{(n)}, \end{aligned}$$

where $\mathbb{Z}[T, T^{-1}]$ is the polynomial subring of rational function field $\mathbb{Q}(T)$ of a variable T over the rational number field.

Since $e_{p,i}\hat{A}e_{q,j} = 0$ for $p+1 < q$ or $q < p$,

$${}_{e_{p,i}A_pe_{p,i}}e_{p,i}\hat{A}e_{p,j}\,{}_{e_{p,j}A_pe_{p,j}} \simeq {}_{e_iAe_i}e_iAe_j\,{}_{e_jAe_j}$$

and

$${}_{e_{p,i}Ae_{p,i}}e_{p,i}\hat{A}e_{p+1,j}\,{}_{e_{p+1,j}Ae_{p+1,j}} \simeq {}_{e_iAe_i}e_iD(A)e_j\,{}_{e_jAe_j},$$

it follows by Lemma 1.2 that

$$\begin{aligned} [e_{p,i}\hat{A}] &= \sum_j c_{ij}(A)[\bar{e}_{p,j}\bar{A}] + \sum_j c_{ij}(D(A))[\bar{e}_{p+1,j}\bar{A}] \\ &= \sum_j c_{ij}(A)[\bar{e}_{p,j}\bar{A}] + \sum_j c_{ji}(A)(f_i/f_j)[\bar{e}_{p+1,j}\bar{A}] \\ &= \sum_j c_{ij}(A)[\bar{e}_{p,j}\bar{A}] + \sum_j c_{ij}(A)(f_i/f_j)\,T[\bar{e}_{p,j}\bar{A}] \end{aligned}$$

in $K_0(\hat{A})$.

Hence the element of $\mathbb{Z}[T, T^{-1}]^{(n)}$ corresponding to the $[e_{p,i}\hat{A}]$ of K_0 (mod $\hat{A}$) is $(0, ..., T^p, ..., 0)(C + F^tCF^{-1}T)$, where T^p is in the ith slot, $C = (c_{ij}(A))_{ij}$, and $F = (\delta_{ij}f_i)_{ij}$.

Consequently we have

THEOREM 1.3. *Let C be the Cartan matrix of A and $C^* = F^tCF^{-1}$, where $F = (\delta_{ij}f_i)_{ij}$. Let Θ be the endomorphism of $\mathbb{Z}[T, T^{-1}]^{(n)}$ obtained by* *the multiplication of $C + C^*T$ from the right hand with elements of $\mathbb{Z}[T, T^{-1}]^{(n)}$. Then it holds that*

$$K_0(\underline{\mathrm{mod}}\,\hat{A}) \simeq \mathbb{Z}[T, T^{-1}]^{(n)}/\mathrm{Im}\,\Theta \simeq \mathrm{Cok}\,\Theta.$$

That is, we have the exact sequence

$$\mathbb{Z}[T, T^{-1}]^{(n)} \xrightarrow{\Theta} \mathbb{Z}[T, T^{-1}]^{(n)} \longrightarrow K_0(\underline{\mathrm{mod}}\,\hat{A}) \longrightarrow 0.$$

2. $K_0(\underline{\mathrm{mod}}\,\hat{A}) \simeq \mathbb{Z}^{(n)}$ AND CARTAN MATRIX OF A

By Theorem 1.3 our main problem is reduced to the following question: When is $\mathbb{Z}[T, T^{-1}]^{(n)}/\mathrm{Im}\,\Theta \simeq \mathbb{Z}^{(n)}$?

THEOREM 2.1. *Let Θ be the endomorphism of $\mathbb{Z}[T, T^{-1}]^{(n)}$ in Theorem 1.3. Then $\mathbb{Q} \otimes_{\mathbb{Z}} (\mathbb{Z}[T, T^{-1}]^{(n)})/\mathrm{Im}\,\Theta \simeq \mathbb{Q}^{(n)}$ if and only if the Cartan matrix C of A is regular, i.e., the determinant of C is non-zero.*

Proof. It is clear that $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z}[T, T^{-1}]^{(n)} \simeq \mathbb{Q}[T, T^{-1}]^{(n)}$ and $\mathbb{Q}[T, T^{-1}] \supset \mathbb{Q}[T]$. Hence we can consider that $C + C^*T$ is an element of the $n \times n$ matrix ring $\mathbb{Q}[T]_n$ over $\mathbb{Q}[T]$. Therefore by elementary transformations we have a diagonal matrix

$$C + C^*T \sim \begin{bmatrix} f_1(T) & & & & & & \\ & f_2(T) & & & & & \\ & & \ddots & & & & \\ & & & f_r(T) & & & \\ & & & & 0 & & \\ & & & & & \ddots & \\ & & & & & & 0 \end{bmatrix},$$

where the $f_i(T)$ are polynomials with non-zero constant term.

Assume that $\mathbb{Q} \otimes_{\mathbb{Z}} (\mathbb{Z}[T, T^{-1}]^{(n)})/\mathrm{Im}\,\Theta \simeq \mathbb{Q}^{(n)}$. In this case it holds that $r = n$, for otherwise, as the elementary transformations are in $\mathbb{Q}[T, T^{-1}]^{(n)}$ we know that $\mathbb{Q}[T, T^{-1}]$ appears as a direct summand of $\mathbb{Q}[T, T^{-1}]/\mathbb{Q} \otimes_{\mathbb{Z}} \mathrm{Im}\,\Theta \simeq \mathbb{Q}^{(n)}$. But $(\mathbb{Q}[T, T^{-1}] : \mathbb{Q}) = \infty$ and this is a contradiction.

Let n_i be the degree of $f_i(T)$, $i = 1, 2, ..., n$. Then it holds that $(\mathbb{Q}[T, T^{-1}]/f_i(T)\mathbb{Q}[T, T^{-1}] : \mathbb{Q}) \leqslant n_i$ since $\{\bar{1}, \bar{T}, ..., \bar{T}^{n_i - 1}\}$ generates $\mathbb{Q}[T, T^{-1}]/f_i(T)\mathbb{Q}[T, T^{-1}]$. Consequently

$$n = (\mathbb{Q}[T, T^{-1}]^{(n)}/\mathbb{Q} \otimes_{\mathbb{Z}} \mathrm{Im}\,\Theta : \mathbb{Q}) \leqslant \sum_{i=1}^{n} n_i.$$

On the other hand it holds that

$$\sum_{i=1}^{n} n_i = \deg\left(\prod_{i=1}^{n} f_i(T)\right) = \deg(\det(C+C^*T)) \leqslant n.$$

This implies $\deg(\det(C + C^*T)) = n$. However, $\det(C + C^*T) = (\det C^*)T^n + \cdots + (\det C)$. Hence $\det C^* \neq 0$. It follows that $\det C \neq 0$ as $C^* = F^tCF^{-1}$ and F is regular.

Assume $\det C \neq 0$. If $\{\bar{1}, \bar{T}, ..., \bar{T}^{n_i-1}\}$ is linearly dependent over $\mathbb{Q}$ in $\mathbb{Q}[T, T^{-1}]/f_i(T)\mathbb{Q}[T, T^{-1}]$, then there are $a_j \in \mathbb{Q}$, $j = 1, 2, ..., n_i - 1$, a positive integer s, and $g(T) \in \mathbb{Q}[T]$ such that $(a_0 + a_1 T + \cdots + a_{n_i} T^{n_i-1}) = f_i(T)\, g(T)/T^s$ and $(g(T), T) = 1$. Then T divides $f_i(T)$. But this contradicts $\det(C + C^*T) = \prod_{i=1}^{n} f_i(T) = \det C + \cdots + \det C^*T^n$ and $\det C \neq 0$. This completes the proof.

COROLLARY 2.2. *Let n be the number of non-isomorphic simple A-modules. Then the rank of the $\mathbb{Z}$-free part of K_0 ($\underline{\text{mod}}\, \hat{A}$) is either $\leqslant n$ or ∞.*

LEMMA 2.3. *Let C be the Cartan matrix of an algebra A and Θ the endomorphism of $\mathbb{Z}[T, T^{-1}]^{(n)}$ in Theorem 1.3. Let $\alpha = \sum_{k=1}^{n} (\sum_{s=p}^{q} a_{k,s} T^s) u_k$ be an element of $\mathbb{Z}[T, T^{-1}]^{(n)}$, where $\{u_k\}$, $k = 1, 2, ..., n$, is a free $\mathbb{Z}[T, T^{-1}]$-basis of $\mathbb{Z}[T, T^{-1}]^{(n)}$ and $\Theta\alpha = \beta = \sum_{k=1}^{n} (\sum_{t=x}^{y} b_{k,t} T^t) u_k$. Let us denote by A_s and B_t the row matrices $(a_{1,s}, a_{2,s}, ..., a_{n,s})$, $s = p, p+1, ..., q$, and $(b_{1,t}, b_{2,t}, ..., b_{n,t})$, $t = x, x+1, ..., y$ in $\mathbb{Z}^{(n)}$, respectively.*

Then it holds that $x = p$ and $y = q + 1$ and $A_p C = B_p$, $A_p C^* + A_{p+1} C = B_{p+1}, ..., A_{q-1} C^* + A_q C = B_q$, $A_q C^* = B_{q+1}$.

Proof. A routine calculation.

PROPOSITION 2.4. *Let* $\det C \neq 0$. *If* $\mathbb{Z}[T, T^{-1}]^{(n)}/\mathrm{Im}\, \Theta$ *is a finitely generated $\mathbb{Z}$-module, then* $\det C = \pm 1$.

Proof. For any $\alpha \in \mathbb{Z}[T, T^{-1}]^{(n)}$ we can put $\alpha = \sum_{k=1}^{n} (\sum_s x_{k,s} T^s) u_k$, where $\{u_k\}$, $k = 1, 2, ..., n$, a free $\mathbb{Z}[T, T^{-1}]$-basis of $\mathbb{Z}[T, T^{-1}]^{(n)}$.

From the assumption we may assume that there are some fixed integers v and m such that $\bar{\alpha} = \sum_{k=1}^{n} (\sum_{s=v}^{m} x_{k,s} \bar{T}^s) \bar{u}_k$. Without loss of generality we may assume further that $v = 0$, since $\bar{T}^{-1}(\mathbb{Z}[T, T^{-1}]^{(n)}/\mathrm{Im}\, \Theta) = \mathbb{Z}[T, T^{-1}]^{(n)}/\mathrm{Im}\, \Theta$. This implies that $\{\bar{T}^s \bar{u}_k\}$, $k = 1, 2, ..., n$, $s = 0, 1, ..., m$, contains a set of generators of $\mathbb{Z}[T, T^{-1}]^{(n)}/\mathrm{Im}\, \Theta$ as a $\mathbb{Z}$-module, and by a similar reason so does $\{\bar{T}^t \bar{u}_k\}$, $k = 1, 2, ..., n$, $t = 1, 2, ..., m+1$.

Now for each i take $\beta_i = (\sum_{k=1}^{n} \delta_{k,i} T^0) u_k \in \mathbb{Z}[T, T^{-1}]^{(n)}$, $i = 1, 2, ..., n$. As quoted above $\{\bar{T}^t \bar{u}_k\}$, $k = 1, 2, ..., n$, $t = 1, 2, ..., m+1$, contains a set of generators of a $\mathbb{Z}$-module $\mathbb{Z}[T, T^{-1}]^{(n)}/\mathrm{Im}\,\Theta$, there exists $\beta'_i = \sum_{k=1}^{n} (\sum_{t=1}^{m+1} b_{k,t}^{(i)} T^t) u_k$ such that $\beta_i + \beta'_i \in \mathrm{Im}\,\Theta$. Hence there exists $\alpha_i \in \mathbb{Z}[T, T^{-1}]^{(n)}$ such that $\Theta\alpha_i = \beta_i + \beta'_i$, $i = 1, 2, ..., n$.

Let us denote α_i by $\sum_{k=1}^{n} (\sum_{s=p}^{q} a_{k,s}^{(i)} T^s) u_k$. Then from the equations in Lemma 2.3 it follows that $p = 0$ and $q = m$, since $\det C = \det C^* \neq 0$. Denote by $A_s^{(i)}$ and $B_t^{(i)}$ the row matrices $(a_{1,s}^{(i)}, a_{2,s}^{(i)}, ..., a_{n,s}^{(i)})$ and $(b_{1,t}^{(i)}, b_{2,t}^{(i)}, ..., b_{n,t}^{(i)})$ in $\mathbb{Z}^{(n)}$, respectively. It holds that $A_m^{(i)} = B_{m+1}^{(i)} C^{*-1}$, $A_{m-1}^{(i)} = (B_m^{(i)} - A_m^{(i)} C) C^{*-1}, ..., A_0^{(i)} = (B_1^{(i)} - A_1^{(i)} C) C^{*-1}$ and $(0, 0, ..., \overset{i}{\check{1}}, ..., 0, 0) = A_0^{(i)} C$. Therefore we have

$$\begin{bmatrix} A_0^{(1)} \\ A_0^{(2)} \\ \vdots \\ A_0^{(n)} \end{bmatrix} C = E.$$

Hence $\det C = \pm 1$ because $A_0^{(i)} \in \mathbb{Z}^{(n)}$, $i = 1, 2, ..., n$.

PROPOSITION 2.5. *Let C be the Cartan matrix of an algebra A. If* $\det C = \pm 1$, *then $\mathbb{Z}[T, T^{-1}]^{(n)}/\mathrm{Im}\,\Theta$ is $\mathbb{Z}$-free.*

Proof. Let β be an element of $\mathbb{Z}[T, T^{-1}]^{(n)}$ such that $d\beta \in \mathrm{Im}\,\Theta$ for an integer d. Let $\alpha = \sum_{k=1}^{n} (\sum_{s=0}^{m} a_{k,s} T^s) u_k$ such that $\Theta\alpha = d\beta$. Denote $\beta = \sum_{k=1}^{n} (\sum_{t=0}^{m+1} b_{k,t} T^t) u_k$, $B_t = (b_{1,t}, b_{2,t}, ..., b_{n,t})$, $t = 0, 1, ..., m+1$, and $A_s = (a_{1,s}, a_{2,s}, ..., a_{n,s})$, $s = 0, 1, ..., m$. Then it holds that $A_0 C = dB_0$, $A_0 C^* + A_1 C = dB_1, ..., A_{m-1} C^* + A_m C = dB_m$, $A_m C^* = dB_{m+1}$. Since $\det C = \pm 1$ and $\det C^* = \pm 1$ there are $A'_s = (a'_{1,s}, a'_{2,s}, ..., a'_{n,s}) \in \mathbb{Z}^{(n)}$ such that $dA'_s = A_s$, $s = 0, 1, ..., m$. Then $\beta = \Theta'\alpha$ for $\alpha' = \sum_{k=1}^{n} (\sum_{s=0}^{m} a'_{k,s} T^s) u_k$. This completes the proof.

Now by Theorem 2.1 and Propositions 2.4 and 2.5 we have

THEOREM 2.6. *Let A be a finite dimensional algebra and $\hat{A}$ its repetitive algebra. Let C be the Cartan matrix of A. Then $K_0\,(\underline{\mathrm{mod}}\,\hat{A}) \simeq K_0\,(\mathrm{mod}\,A)$ if and only if* $\det C = \pm 1$.

In the following examples Q's are the quivers of algebras A's and $(*)$'s are their relations, respectively. Of course C and $K_0\,(\underline{\mathrm{mod}}\,\hat{A})$ are their Cartan matrices and stable Grothendieck groups.

EXAMPLE 1 (cf. [1]).

Q: $\alpha \quad 1 \underset{\gamma}{\overset{\beta}{\rightleftarrows}} 2$, $\quad (*)$: $\alpha^2 = 0$, $\beta\alpha = 0$, $\gamma\beta = 0$, $\beta\gamma = 0$, $\alpha\gamma = 0$.

Then

$$C=\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

and $K_0\,(\underline{\mathrm{mod}}\,\hat{A}) \simeq \mathbb{Z} \oplus \mathbb{Z} \simeq K_0(A)$, but gl. dim $A = \infty$.

EXAMPLE 2.

$$Q\colon 1 \xrightarrow{\alpha} 2 \underset{\gamma}{\overset{\beta}{\rightleftarrows}} 3, \qquad (*)\colon \beta\alpha = 0,\ \beta\gamma = 0,\ \gamma\beta = 0,$$

Then

$$C=\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

and $K_0\,(\underline{\mathrm{mod}}\,\hat{A}) \simeq \mathbb{Z} \not\simeq K_0(A)$. Therefore it occurs that $K_0\,(\underline{\mathrm{mod}}\,\hat{A})$ is a finitely generated free $\mathbb{Z}$-module, but not isomorphic to $K_0(A)$.

EXAMPLE 3.

$$Q\colon\ \begin{tikzcd} 1 \arrow[loop left, "\sigma"] \arrow[r, shift left=3, "\alpha_1"] \arrow[r, shift left=1, "\alpha_2"'] & 2 \arrow[l, shift left=1, "\beta_1"] \arrow[l, shift left=3, "\beta_2"] \arrow[loop right, "\rho"] \end{tikzcd},$$

$(*)$: $\sigma^2=0$, $\rho^3=0$, $\alpha_i\sigma=0$, $\sigma\beta_i=0$, $\alpha_i\beta_j=0$, $\beta_i\alpha_j=0$, $\rho\alpha_i=0$, $\beta_i\rho=0$ for $i, j=1, 2$. Then

$$C=\begin{bmatrix} 2 & 2 \\ 2 & 3 \end{bmatrix}$$

and $\det C = 2$, hence $\mathbb{Q} \otimes K_0\,(\underline{\mathrm{mod}}\,\hat{A}) \simeq \mathbb{Q} \otimes K_0(A)$, but $K_0\,(\underline{\mathrm{mod}}\,\hat{A}) \simeq \mathbb{Z}^{(2)} \oplus (\bigoplus_{\mathbb{Z}} \mathbb{Z}/(2))$.

EXAMPLE 4.

$$Q\colon\ \begin{tikzcd} 1 \arrow[r, shift left=3, "\alpha_1"] \arrow[r, shift left=1, "\alpha_2"'] & 2 \arrow[l, shift left=1, "\beta_1"] \arrow[l, shift left=3, "\beta_2"] \arrow[loop right, "\rho"] \end{tikzcd},$$

$(*)$: $\rho^4=0$, $\rho\alpha_i=0$, $\beta_j\rho=0$, $\alpha_i\beta_j=0$, $\beta_j\alpha_i=0$ for $i, j=1, 2$. Then

$$C=\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$$

and $\det C=0$, $K_0\,(\underline{\mathrm{mod}}\,\hat{A})\simeq\mathbb{Z}\oplus\mathbb{Z}[T, T^{-1}]$. Hence for this example $K_0\,(\underline{\mathrm{mod}}\,\hat{A})$ is $\mathbb{Z}$-free, but not finitely generated.

EXAMPLE 5.

$$Q:\quad \begin{tikzcd}[column sep=large] 1 \arrow[r, shift left=6, "\alpha_1"] \arrow[r, shift left=3, "\alpha_2"] \arrow[r, "\alpha_3"] & 2 \arrow[l, shift left=3, "\beta_1"] \arrow[l, shift left=6, "\beta_2"] \arrow[loop right, "\rho"] \end{tikzcd},$$

$(*)$: $\rho^6=0$, $\rho\alpha_i=0$, $\beta_j\rho=0$, $\alpha_i\beta_j=0$, $\beta_j\alpha_i=0$ for $i=1, 2, 3$, $j=1, 2$. Then

$$C=\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$$

and $\det C=0$, $K_0\,(\underline{\mathrm{mod}}\,\hat{A})\simeq 0$.

REFERENCES

1. W. D. BURGESS, K. R. FULLER, E. R. VOSS, AND B. ZIMMERMANN-HUISGEN, The Cartan matrix as an indicator of finite global dimension for artinian rings, *Proc. Amer. Math. Soc.* **95** (1985), 157–165.
2. D. HAPPEL, On the derived category of a finite dimensional algebra, *Comment. Math. Helv.* **62** (1987), 339–389.
3. D. HAPPEL, Triangulated categories in the representation theory of finite dimensional algebras, *in* "London Math. Soc. Lecture Note Ser., Vol. 119," Cambridge Univ. Press, Cambridge/New York, 1988.
4. D. HUGHES AND J. WASCHBUSCH, Trivial extensions of tilted algebras, *Proc. London Math. Soc.* **46** (1983), 347–364.
5. G. V. WILSON, The Cartan map on categories of graded modules, *J. Algebra* **85** (1983), 390–398.